August 1996
WATPHYS-TH-96/11
hep-th/9609085

Quantum scalar field on three-dimensional (BTZ) black hole instanton: heat kernel, effective action and thermodynamics

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Abstract

We consider the behaviour of a quantum scalar field on three-dimensional Euclidean backgrounds: Anti-de Sitter space, the regular BTZ black hole instanton and the BTZ instanton with a conical singularity at the horizon. The corresponding heat kernel and effective action are calculated explicitly for both rotating and non-rotating holes. The quantum entropy of the BTZ black hole is calculated by differentiating the effective action with respect to the angular deficit at the conical singularity. The renormalization of the UV-divergent terms in the action and entropy is considered. The structure of the UV-finite term in the quantum entropy is of particular interest. Being negligible for large outer horizon area A_+ it behaves logarithmically for small A_+. Such behaviour might be important at late stages of black hole evaporation.

PACS number(s): 04.70.Dy, 04.62.+v

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1 Introduction

Early interest in lower-dimensional black hole physics [1] has grown into a rich and fruitful field of research. The main motivation for this is that the salient problems of quantum black holes, such as loss of information and the endpoint of quantum evaporation, can be more easily understood in some simple low-dimensional models than directly in four dimensions [2]. Several interesting 2D candidates have been explored to this end which share many common features with their four-dimensional cousins [3]. This is intriguing since the one-loop quantum effective action in two dimensions is exactly known, in the form of the Polyakov-Liouville term, giving rise to the hope that the semiclassical treatment of quantum black holes in two dimensions can be done explicitly (see reviews [2]).

The black hole in three-dimensional gravity discovered by Bañados, Teitelboim and Zanelli (BTZ) [4] has features that are even more realistic than its two-dimensional counterparts. It is similar to the Kerr black hole, being characterized by mass M and angular momentum J and having an event horizon and (for $J \neq 0$) an inner horizon [5, 6, 7, 8]. This solution naturally appears as the final stage of collapsing matter [9]. In contrast to the Kerr solution it is asymptotically Anti-de Sitter rather than asymptotically flat. Geometrically, the BTZ black hole is obtained from 3D Anti-de Sitter (AdS_3) spacetime by performing some identifications. Although quantum field theory on curved three-dimensional manifolds is not as well understood as in two dimensions, the large symmetry of the BTZ geometry and its relation to AdS_3 allow one to obtain some precise results when field is quantized on this background. The Green's function and quantum stress tensor for the conformally coupled scalar field and the resultant back reaction were calculated in [10, 11, 12].

The possibility that black hole entropy might have a statistical explanation remains an intriguing issue, and there has been much recent activity towards its resolution via a variety of approaches (for a review see [13]). One such proposal is that the Bekenstein-Hawking entropy is completely generated by quantum fields propagating in the black hole background. Originally it was belived that UV-divergent quantum corrections associated with such fields to the Bekenstein-Hawking expression play a fundamental role in the

statistical interpretation of the entropy. However, it was subsequently realized that these divergent corrections can be associated with those that arise from the standard UV-renormalization of the gravitational couplings in the effective action [14], [15], [16], [18], [17], [19]. The idea of complete generation of the entropy by quantum matter in the spirit of induced gravity [15] encountered the problem of an appropriate statistical treatment of the entropy of non-minimally-coupled matter [20], [21] (see, however, another realization of this idea within the superstring paradigm [22]). At the same time, it was argued in number of papers [23, 24, 25, 26] that UV-finite quantum corrections to the Bekenstein-Hawking entropy might be even more important than the UV-infinite ones. They could provide essential modifications of the thermodynamics of a hole at late stages of the evaporation when quantum effects come to play.

Relatively little work has been done concerning the quantum aspects of the entropy for the BTZ black hole. Carlip [27] has shown that the appropriate quantization of 3D gravity represented in the Chern-Simons form yields a set of boundary states at the horizon. These can be counted using methods of Wess-Zumino-Witten theory. Remarkably, the logarithm of their number gives the classical Bekenstein-Hawking formula. This is the unique case of a statistical explanation of black hole entropy. Unfortunately, it is essentially based on features peculiar to three-dimensional gravity and its extension to four dimensions is not straightforward. An investigation of the thermodynamics of quantum scalar fields on the BTZ background [28] concluded that the divergent terms in the entropy are not always due to the existence of the outer horizon (i.e. the leading term in the quantum entropy is not proportional to the area of the outer horizon) and depend upon the regularization method. This conclusion seems to be in disagreement with the expectations based on the study of the problem in two and four dimensions.

In this paper we systematically calculate the heat kernel, effective action and quantum entropy of scalar matter for the BTZ black hole. The relevant operator is $(\Box + \xi/l^2)$, where ξ is an arbitrary constant and $1/l^2$ is the cosmological constant appearing in the BTZ solution. Since we are interested in the thermodynamic aspects we consider the Euclidean BTZ geometry with a conical singularity at the horizon as the background. In the process of getting the heat kernel and effective action on this singular geometry we

proceed in steps, first calculating quantities explicitly for AdS_3, then the regular BTZ instanton and finally the conical BTZ instanton. The entropy is calculated by differentiating the effective action with respect to the angular deficit at the horizon. It contains both UV-divergent and UV-finite terms. The analysis of the divergences shows that they are explicitly renormalized by renormalization of Newton's constant in accordance with general arguments [18].

We find the structure of the UV-finite terms in the entropy to be particularly interesting. These terms, negligible for large outer horizon area A_+, behave logarithmically at small A_+. Hence they should become important at late stages of black hole evaporation.

The paper is organized as follows. In Section 2 we briefly review the Euclidean BTZ geometry, omitting details that appear in earlier work [4, 5, 6, 7, 8]. We discuss in section 3 various forms of the metric for 3D Anti-de Sitter space giving expressions for the geodesic distance on AdS_3 that are relevant for our purposes. We solve explicitly the heat kernel equation and find the Green's function on AdS_3 as a function of the geodesic distance. In Section 4 we calculate explicitly the trace of heat kernel and the effective action on the regular and singular Euclidean BTZ instantons. The quantum entropy is the subject of Section 5 and in Section 6 we provide some concluding remarks.

2 Sketch of BTZ black hole geometry

We start with the black hole metric written in a form that makes it similar to the four-dimensional Kerr metric. Since we are interested in its thermodynamic behaviour, we write the metric in the Euclidean form:

$$ds^2 = f(r)d\tau^2 + f^{-1}(r)dr^2 + r^2(d\phi + N(r)d\tau)^2 \quad , \tag{2.1}$$

where the metric function $f(r)$ reads

$$f(r) = \frac{r^2}{l^2} - \frac{j^2}{r^2} - m = \frac{(r^2 - r_+^2)(r^2 + |r_-|^2)}{l^2 r^2} \tag{2.2}$$

and we use the notation

$$r_+^2 = \frac{ml^2}{2}(1 + \sqrt{1 + (\frac{2j}{ml})^2}) \ , \quad |r_-|^2 = \frac{ml^2}{2}(\sqrt{1 + (\frac{2j}{ml})^2} - 1) \tag{2.3}$$

where we note the useful identity

$$r_+|r_-| = jl$$

for future reference. The function $N(r)$ in (2.1) is

$$N(r) = -\frac{j}{r^2} \quad . \tag{2.4}$$

In order to transform the metric (2.1) to Lorentzian signature we need to make the transformation: $\tau \to \imath\, t$, $j \to -\imath\, j$. Then we have that

$$\begin{aligned} r_+ \to r_+^L &= \sqrt{\frac{ml^2}{2}} \left(1 + \sqrt{1 - (\frac{2j}{ml})^2}\right)^{1/2} \quad , \\ |r_-| \to \imath\, r_-^L &= \sqrt{\frac{ml^2}{2}} \left(1 - \sqrt{1 - (\frac{2j}{ml})^2}\right)^{1/2} \quad , \end{aligned} \tag{2.5}$$

where r_+^L and r_-^L are the values in the Lorentzian space-time. These are the respective radii of the outer and inner horizons of the Lorentzian black hole in $(2+1)$ dimensions. Therefore we must always apply the transformation (2.5) after carrying out all calculations in the Euclidean geometry in order to obtain the result for the Lorentzian black hole. The Lorentzian version of the metric (2.1) describes a black hole with mass m and angular momentum $J = 2j$ [4], [5]. Introducing

$$\beta_H \equiv \frac{2}{f'(r_+)} = \frac{r_+ l^2}{r_+^2 + |r_-|^2} \tag{2.6}$$

we find that in the (τ, r) sector of the metric (2.1) there is no conical singularity at the horizon if the Euclidean time τ is periodic with period $2\pi\beta_H$. The quantity $T_H = (2\pi\beta_H)^{-1}$ is the Hawking temperature of the hole.

The horizon Σ is a one-dimensional space with metric $ds_\Sigma^2 = l^2 d\psi^2$, where $\psi = \frac{r_+}{l}\phi - \frac{|r_-|}{l^2}\tau$ is a natural coordinate on the horizon.

Looking at the metric (2.1) one can conclude that there is no constraint on the periodicity of the "angle" variable ϕ (or ψ). This is in contrast to the four-dimensional black hole, for which the angle ϕ in the spherical line element $(d\theta^2 + \sin^2\theta d\phi^2)$ varies between the limits $0 \le \phi \le 2\pi$ in order to avoid the appearance of the conical singularities at the poles of the sphere. However, following tradition we will assume that the metric (2.1) is periodic in ϕ, with limits $0 \le \phi \le 2\pi$. This means that Σ is a circle with length ("area") $A_+ = 2\pi r_+$.

There are a number of other useful forms for the metric (2.1). It is very important for our considerations that (2.1) is obtained from the metric of three-dimensional Anti-de Sitter space by making certain identifications along the trajectories of its Killing vectors. In order to find the appropriate metric for the 3D Anti-de Sitter space (denoted below by H_3) we consider a four-dimensional flat space with metric

$$ds^2 = dX_1^2 - dT_1^2 + dX_2^2 + dT_2^2 \quad . \tag{2.7}$$

AdS$_3$ (H_3) is defined as a subspace defined by the equation

$$X_1^2 - T_1^2 + X_2^2 + T_2^2 = -l^2 \quad . \tag{2.8}$$

Introducing the coordinates (ψ, θ, χ)

$$\begin{aligned} X_1 &= \frac{l}{\cos\chi} \sinh\psi \ , \quad T_1 = \frac{l}{\cos\chi} \cosh\psi \\ X_2 &= l \ \tan\chi \cos\theta \ , \quad T_2 = l \ \tan\chi \sin\theta \end{aligned} \tag{2.9}$$

the metric on H_3 reads

$$ds^2_{H_3} = \frac{l^2}{\cos^2\chi}(d\psi^2 + d\chi^2 + \sin^2\chi d\theta^2) \quad . \tag{2.10}$$

It is easy to see that under the coordinate transformation

$$\begin{aligned} \psi &= \frac{r_+}{l}\phi - \frac{|r_-|}{l^2}\tau \ , \quad \theta = \frac{r_+}{l}\tau + \frac{|r_-|}{l^2}\phi \\ \cos\chi &= (\frac{r_+^2 + |r_-|^2}{r^2 + |r_-|^2})^{1/2} \end{aligned} \tag{2.11}$$

the metric (2.1) coincides with (2.10). In the next section we will derive a few other forms of the metric on H_3 which are useful in the context of calculation of the heat kernel and Green's function on H_3.

The BTZ black hole (B_3) described by the metric (2.1) is obtained from AdS$_3$ with metric (2.10) by making the following identifications:

i). $(\psi, \theta, \chi) \rightarrow (\psi, \theta + 2\pi, \chi)$. This means that $(\phi, \tau, r) \rightarrow (\phi + \Phi, \tau + T_H^{-1}, r)$, where $\Phi = T_H^{-1} j r_+^{-2}$.

ii). $(\psi, \theta, \chi) \rightarrow (\psi + 2\pi\frac{r_+}{l}, \theta + 2\pi\frac{|r_-|}{l}, \chi)$, which is the analog of $(\phi, \tau, r) \rightarrow (\phi + 2\pi, \tau, r)$.

The coordinate χ is the analog of the radial coordinate r. It has the range $0 \leq \chi \leq \frac{\pi}{2}$. The point $\chi = 0$ is the horizon ($r = r_+$) while $\chi = \frac{\pi}{2}$ lies at infinity. Geometrically,

i) means that there is no conical singularity at the horizon, which is easily seen from (2.10). A section of BTZ black hole at fixed χ is illustrated in Fig.1 for the non-rotating ($|r_-| = 0$) and rotating cases. The opposite sides of the quadrangle in Fig.1 are identified. Therefore, the whole section looks like a torus. In the rotating case the torus is deformed with deformation parameter γ, where $\tan\gamma = \frac{r_+}{|r_-|}$. The whole space B_3 is a region between two semispheres with $R = \exp(\psi)$ being radius, χ playing the role of azimuthal angle and θ being the orbital angle. The boundaries of the region are identified according to ii).

3 3D Anti-de Sitter space: geometry, heat kernel and Green's function

3.1 Metric on H_3

3D Anti-de Sitter space (H_3) is defined as a 3-dimensional subspace of the flat four-dimensional space-time with metric

$$ds^2 = dX_1^2 - dT_1^2 + dX_2^2 + dT_2^2 \tag{3.1}$$

satisfying the constraint

$$X_1^2 - T_1^2 + X_2^2 + T_2^2 = -l^2 \quad . \tag{3.2}$$

We are interested in AdS$_3$, which has Euclidean signature. This is easily done by appropriately choosing the signature in (3.1), (3.2). The induced metric has a number of different representations depending on the choice of the coordinates on AdS$_3$. Below we consider two such choices.

A. Resolve equation (3.2) as follows:

$$\begin{aligned} X_1 &= l\cosh\rho\sinh\psi\ , \quad T_1 = l\cosh\rho\cosh\psi \\ X_2 &= l\sinh\rho\cos\theta\ , \quad T_1 = l\sinh\rho\sin\theta \quad . \end{aligned} \tag{3.3}$$

The variables (ρ, ψ, θ) can be considered as coordinates on H_3. They are closely related to the system (χ, ψ, θ) via the transformation $\cos\chi = \cosh^{-1}\rho$. Note that the section of H_3 corresponding to a fixed ρ is a two-dimensional torus. The induced metric then takes the following form:

$$ds^2_{H_3} = l^2\left(d\rho^2 + \cosh^2\rho d\psi^2 + \sinh^2\rho d\theta^2\right) \quad . \tag{3.4}$$

The BTZ black hole metric is then obtained from (3.4) by making the identifications $\theta \to \theta + 2\pi$ and $\psi \to \psi + 2\pi \frac{r_+}{l}$, $\theta \to \theta + 2\pi \frac{|r_-|}{l}$.

B. Another way to resolve the constraint (3.2) is by employing the transformation

$$\begin{aligned} X_1 &= l \sinh(\sigma/l) \cos\lambda\ , \quad T_1 = l\cosh(\sigma/l) \\ X_2 &= l \sinh(\sigma/l) \sin\lambda \sin\phi\ , \quad T_2 = l \sinh(\sigma/l) \sin\lambda \cos\phi\ \ . \end{aligned} \tag{3.5}$$

The section $\sigma = const$ of H_3 is a two-dimensional sphere. The induced metric in the coordinates (σ, λ, ϕ) takes the form

$$ds^2_{H^3} = d\sigma^2 + l^2 \sinh^2(\sigma/l)(d\lambda^2 + \sin^2\lambda d\phi^2) \tag{3.6}$$

from which one can easily see that H_3 is a hyperbolic version of the metric on the 3-sphere

$$ds^2_{S^3} = d\sigma^2 + l^2 \sin^2(\sigma/l)(d\lambda^2 + \sin^2\lambda d\phi^2)\ \ , \tag{3.7}$$

allowing us to making use of our experience with the 3-sphere in understanding the geometry of H_3.

3.2 Geodesic distance on H_3

An important fact equally applicable both to S_3 and H_3 is the following. Consider two different points on S_3 (H_3). Then we can choose the coordinate system (σ, λ, ϕ) such that one of the points lies at the origin ($\sigma = 0$) and the other point lies on the radius $(\sigma, \lambda = 0, \phi)$. This radial trajectory joining the two points is a geodesic. Moreover, the geodesic distance between these two points coincides with σ. More generally, for the metric (3.6), (3.7) the geodesic distance between two points with equal values of λ and ϕ ($\lambda = \lambda'\ ,\ \ \phi = \phi'$) is given by $|\sigma - \sigma'| = \Delta\sigma$.

In order to find the geodesic distance in the coordinate system (ρ, ψ, θ) (3.3) consider the following trick. The two points M and M' in the embedding four-dimensional space determine the vectors $\mathbf{a}$ and $\mathbf{a}'$ starting from the origin:

$$\begin{aligned} \mathbf{a} &= l\cosh\rho \sinh\psi\ \mathbf{x_1} + l\cosh\rho\cosh\psi\ \mathbf{t_1} + l\sinh\rho\cos\theta\ \mathbf{x_2} + l\sinh\rho\sin\theta\ \mathbf{t_2} \\ \mathbf{a}' &= l\cosh\rho' \sinh\psi'\ \mathbf{x_1} + l\cosh\rho'\cosh\psi'\ \mathbf{t_1} + l\sinh\rho'\cos\theta'\ \mathbf{x_2} + l\sinh\rho'\sin\theta'\ \mathbf{t_2}\ \ , \end{aligned} \tag{3.8}$$

where $(\mathbf{t_1}, \mathbf{x_1}, \mathbf{t_2}, \mathbf{x_2})$ is an orthonormal basis of vectors in the space (3.1):

$$-(\mathbf{t_1}, \mathbf{t_1}) = (\mathbf{x_1}, \mathbf{x_1}) = (\mathbf{t_2}, \mathbf{t_2}) = (\mathbf{x_2}, \mathbf{x_2}) = 1 \quad . \tag{3.9}$$

For the scalar product of $\mathbf{a}$ and $\mathbf{a}'$ we have

$$(\mathbf{a}, \mathbf{a}') = l^2 \left(-\cosh^2 \rho \cosh \Delta\psi + \sinh^2 \rho \cos \Delta\theta \right) \quad , \tag{3.10}$$

where $\Delta\psi = \psi - \psi'$, $\Delta\theta = \theta - \theta'$ and for simplicity we assumed that $\rho = \rho'$. The scalar product $(\mathbf{a}, \mathbf{a}')$ is invariant quantity not dependent on a concrete choice of coordinates. Therefore, we can calculate it in the coordinate system (σ, λ, ϕ). In this system we have

$$\begin{aligned} \mathbf{a} &= l \cosh(\sigma/l) \ \mathbf{t}'_\mathbf{1} + l \cosh(\sigma/l) \ \mathbf{x}'_\mathbf{1} \\ \mathbf{a}' &= l \cosh(\sigma'/l)' \ \mathbf{t}'_\mathbf{1} + l \cosh(\sigma'/l) \ \mathbf{x}'_\mathbf{1} \quad . \end{aligned} \tag{3.11}$$

The new basis $(\mathbf{t}'_\mathbf{1}, \mathbf{x}'_\mathbf{1}, \mathbf{t}'_\mathbf{2}, \mathbf{x}'_\mathbf{2})$ is obtained from the old basis $(\mathbf{t_1}, \mathbf{x_1}, \mathbf{t_2}, \mathbf{x_2})$ by some orhogonal rotation. Therefore, it satisfies the same identities (3.9). In new basis we have

$$(\mathbf{a}, \mathbf{a}') = -l^2 \cosh \frac{\Delta\sigma}{l} \quad . \tag{3.12}$$

As we explained above $\Delta\sigma$ is the geodesic distance between M and M'. Equating (3.10) and (3.12) we finally obtain the expression for the geodesic distance in terms of the coordinates (ρ, ψ, θ):

$$\cosh \frac{\Delta\sigma}{l} = \cosh^2 \rho \cosh \Delta\psi - \sinh^2 \rho \cos \Delta\theta \tag{3.13}$$

or alternatively, after some short manipulations

$$\sinh^2 \frac{\Delta\sigma}{2l} = \cosh^2 \rho \sinh^2 \frac{\Delta\psi}{2} + \sinh^2 \rho \sin^2 \frac{\Delta\theta}{2} \quad . \tag{3.14}$$

For small $\rho << 1$ and $\Delta\psi << 1$ from (3.14) we get

$$\Delta\sigma^2 = l^2 \left(\Delta\psi^2 + 4\rho^2 \sin^2 \frac{\Delta\theta}{2} \right) \tag{3.15}$$

what coincides with the result for 3D flat space in cylindrical coordinates.

Note, that $\Delta\sigma$ in (3.13), (3.14) is the intrinsic geodesic distance on H_3. It is worth comparing with the chordal four-dimensional distance Σ between the points M and M'

measured in the imbedding 4D space. In the coordinate system (3.3) we obtain

$$\Sigma^2 \equiv \sum (X - X')^2 = l^2(\cosh^2\rho(\sinh\psi - \sinh\psi')^2 - \cosh^2\rho(\cosh\psi - \cosh\psi')^2$$
$$+\sinh^2\rho(\cos\theta - \cos\theta')^2 + \sinh^2\rho(\sin\theta - \sin\theta')^2) \quad . \tag{3.16}$$

After simplification we obtain

$$\Sigma^2 = 4l^2 \sinh^2 \frac{\Delta\sigma}{2l} \quad . \tag{3.17}$$

Consider now the point M'' which is antipodal to the point M'. It is obtained from M' by antipodal transformation $X' \rightarrow -X'$ (in the coordinates (χ, θ, ψ) the antipode has coordinates $(\pi - \chi, \theta, \psi)$). The point M'' lies in the lower "semisphere" of the space H_3. For some applications we will need the chordal distance $\hat{\Sigma}$ bewteen points M and M'': $\hat{\Sigma}^2 = \sum (X + X')^2$, where we find

$$\hat{\Sigma}^2 = -4l^2 \cosh^2 \frac{\Delta\sigma}{2l} \quad . \tag{3.18}$$

Here $\Delta\sigma$ is the geodesic distance between M and M'.

3.3 Heat kernel and Green's function

Consider on H_3 the heat kernel equation

$$(\partial_s - \Box - \xi/l^2)K(x, x', s) = 0$$
$$K(x, x', s = 0) = \delta(x, x') \quad , \tag{3.19}$$

where s is a proper time variable. The operator $(\Box + \xi/l^2)$ on H_3 or B_3 can be equivalently represented in the form of non-minimal coupling $(\Box - \frac{\xi}{6}R)$. For $\xi = \frac{3}{4}$ this operator would be conformal invariant. This equivalence, however, is no longer valid for the space B_3^α which has a conical singularity. This is because the scalar curvature on a conical space has a δ-function-like contribution due to a singularity that is additional to the regular value of the curvature. The δ-function in the operator $(\Box - \frac{\xi}{6}R)$ has been shown [17] to non-trivially modify the regular heat kernel. In order to avoid the problem of dealing with this perculiarity we will not make use of this form of the operator and will treat the term ξ/l^2 as just a constant that is unrelated to the curvature of space-time.

The function $K(x,x',s)$ satisfying (3.19) can be found as some function of the geodesic distance σ between the points x and x'. The simplest way to do this is to use the coordinate system (σ,λ,ϕ) with the metric (3.6) when both points lie on the radius: $\lambda=\lambda',\phi=\phi'$. Then the Laplace operator $\Box=\nabla^\mu\nabla_\mu$ has only the "radial" part:

$$\Box=\frac{1}{l^2\sinh^2\frac{\sigma}{l}}\partial_\sigma\sinh^2(\frac{\sigma}{l})\partial_\sigma=\frac{1}{l^2\sinh\frac{\sigma}{l}}\partial^2_\sigma\sinh(\frac{\sigma}{l})-l^{-2} \quad . \tag{3.20}$$

Equation (3.19) is then easily solved and the solution takes the form

$$K_{H_3}(\sigma,s)=\frac{1}{(4\pi s)^{3/2}}\frac{\sigma/l}{\sinh(\sigma/l)}e^{-\frac{\sigma^2}{4s}-\mu\frac{s}{l^2}} \quad , \tag{3.21}$$

where $\mu=1-\xi$.

In the conformal case we have $\xi=3/4$ and $\mu=1/4$. The heat kernel (3.21) was first found by Dowker and Critchley [29] for S_3 (for which $\sinh(\sigma/l)$ is replaced by $\sin(\sigma/l)$) and then was extended to the hyperbolic space H_3 by Camporesi [30].

Knowing the heat kernel function $K(\sigma,s)$ we can find the Green's function $G(x,x')$ as follows

$$G(x,x')=\int_0^{+\infty}ds\ K(x,x',s) \quad .$$

Applying this to the heat kernel (3.21) and using the integral

$$\int_0^\infty\frac{ds}{s^{3/2}}e^{-bs-\frac{a^2}{4s}}=\frac{2\sqrt{\pi}}{a}\left(\cosh(\sqrt{b}a)-\sinh(\sqrt{b}a)\right) \tag{3.22}$$

the Green's function on H_3 reads

$$G_{H_3}(x,x')=\frac{1}{4\pi}\frac{1}{l\sinh(\frac{\sigma}{l})}\left(\cosh(\sqrt{\mu}\frac{\sigma}{l})-\sinh(\sqrt{\mu}\frac{\sigma}{l})\right) \quad , \tag{3.23}$$

where σ is the instrinsic geodesic distance on H_3 between x and x'. It is important to observe that the function $G_{H_3}(x,x')$ vanishes when $\cosh(\sqrt{\mu}\frac{\sigma}{l})=\sinh(\sqrt{\mu}\frac{\sigma}{l})$. This happens when $\sigma(x,x')=\infty$, i.e. one of the points lies on the equator ($\chi=\frac{\pi}{2}$). This fact is important in view of the arguments of [31] that the correct quantization on a non-globally hyperbolic space, like AdS_3, requires the fixing of some boundary condition for a quantum field at infinity. The Green's function (3.23) constructed by means of the heat kernel (3.21) automatically satisfies the Dirichlet boundary condition and thus provides for us the correct quantization on H_3. To our knowledge, the form (3.23) of the Green's function on H_3 is not known in the current literature.

A special case occurs when $\xi = 3/4$ and $\mu = 1/4$, for which the operator $(\Box + \xi/l^2) \equiv (\Box - \frac{1}{8}R)$ is conformally invariant. In this case we get

$$G_{H_3} = \frac{1}{4\pi}\left(\frac{1}{2l\sinh\frac{\sigma}{2l}} - \frac{1}{2l\cosh\frac{\sigma}{2l}}\right) \tag{3.24}$$

for the Green's function. Using (3.16) and (3.17) we observe that (3.24) has a nice form in terms of the chordal distance in the imbedding space:

$$G_{H_3}(x,x') = \frac{1}{4\pi}\left(\frac{1}{|\Sigma|} - \frac{1}{|\hat{\Sigma}|}\right) = \frac{1}{4\pi}\left(\frac{1}{|X-X'|} - \frac{1}{|X+X'|}\right) \quad . \tag{3.25}$$

The Green's function for the conformal case in the form (3.25) was reported by Steif [10].

4 Heat kernel on the Euclidean BTZ instanton

4.1 Regular BTZ instanton

As was explained in Section 2 the regular Euclidean BTZ instanton (B_3) may be obtained from H_3 by a combination of identifications which in the coordinates (ρ,θ,ψ) are

$i)$. $\theta \to \theta + 2\pi$

$ii)$. $\theta \to \theta + 2\pi\frac{|r_-|}{l}$, $\psi \to \psi + 2\pi\frac{r_+}{l}$

Therefore, the heat kernel K_{B_3} on the BTZ instanton B_3 is constructed via the heat kernel K_{H_3} on H_3 as infinite sum over images

$$K_{B_3}(x,x',s) = \sum_{n=-\infty}^{+\infty} K_{H_3}(\rho,\ \rho',\ \psi-\psi'+2\pi\frac{r_+}{l}n,\ \theta-\theta'+2\pi\frac{|r_-|}{l}n) \quad . \tag{4.1}$$

Using the path integral representation of heat kernel we would say that the $n=0$ term in (4.1) is due to the direct way of connecting points x and x' in the path integral. On the other hand, the $n \neq 0$ terms are due to uncontractible winding paths that go n times around the circle. Note that K_{H_3} automatically has the periodicity given in $i)$. Therefore the sum over images in (4.1) provides us with the periodicity $ii)$. Assuming that $\rho = \rho'$ it can be represented in the form

$$\begin{aligned} K_{B_3} &= \sum_{n=-\infty}^{\infty} K_{H_3}(\sigma_n, s) \ , \\ \cosh\frac{\sigma_n}{l} &= \cosh^2\rho\cosh\Delta\psi_n - \sinh^2\rho\cos\Delta\theta_n \ , \\ \Delta\psi_n &= \psi-\psi'+2\pi\frac{r_+}{l}n \ , \quad \Delta\theta_n = \theta-\theta'+2\pi\frac{|r_-|}{l}n \ , \end{aligned} \tag{4.2}$$

where $K_{H_3}(\sigma, s)$ takes the form (3.20).

For the further applications consider the integral

$$Tr_w K_{B_3} \equiv \int_{B_3} K_{B_3}(\rho = \rho', \psi = \psi', \theta = \theta' + w)\ d\mu_x \quad , \tag{4.3}$$

where $d\mu_x = l^3 \cosh\rho \sinh\rho d\rho\theta d\psi$ is the measure on B_3. Note that volume of B_3

$$V_{B_3} = \int_{B_3} d\mu_x = l^3 \int_0^{2\pi} d\theta \int_0^{\frac{2\pi r_+}{l}} d\psi \int_0^{+\infty} \cosh\rho \sinh\rho d\rho$$

is infinite and so does not depend on $|r_-|$. This is just a simple consequence of the geometrical fact that the two quadrangles in Fig.1 have the same area.

The integration in (4.3) can be easily performed if for a fixed n we change the variable $\rho \to \bar{\sigma}_n = \sigma_n / l$ (see Eqs.(4.2), (3.13)) with the corresponding change of integration measure

$$\cosh\rho \sinh\rho d\rho = \frac{1}{2} \frac{\sinh\bar{\sigma}_n d\bar{\sigma}_n}{(\cosh\Delta\psi_n - \cos\Delta\theta_n)} = \frac{1}{4} \frac{\sinh\bar{\sigma}_n d\bar{\sigma}_n}{(\sinh^2\frac{\Delta\psi_n}{2} + \sin^2\frac{\Delta\theta_n}{2})} \quad .$$

Then after integration Eq.(4.3) reads

$$\begin{aligned} Tr_w K_{B_3} &= V_w\ \frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}} + (2\pi)\ (\frac{2\pi r_+}{l})\ \frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}\ \bar{s} \sum_{n=1}^{\infty} \frac{e^{-\frac{\Delta\psi_n^2}{4\bar{s}}}}{(\sinh^2\frac{\Delta\psi_n}{2} + \sin^2\frac{\Delta\theta_n}{2})} \quad , \\ V_w &= \begin{cases} \frac{V_{B_3}}{l^3} & \text{if } w = 0 \quad , \\ (2\pi)(\frac{2\pi r_+}{l}) \frac{1}{\sin^2\frac{w}{2}} \frac{\bar{s}}{2} & \text{if } w \neq 0 \quad , \end{cases} \end{aligned} \tag{4.4}$$

where we defined $\bar{s} = s/l^2$, $\Delta\psi_n = \frac{2\pi r_+}{l} n$, $\Delta\theta_n = w + \frac{2\pi|r_-|}{l} n.$

The knowledge of the heat kernel allows us to calculate the effective action on B_3:

$$\begin{aligned} W_{eff}[B_3] &= -\frac{1}{2} \int_{\epsilon^2}^{\infty} \frac{ds}{s} Tr_{w=0} K_{B_3} \\ &= W_{div}[B_3] - \sum_{n=1}^{\infty} \frac{1}{4n} \frac{e^{-\sqrt{\mu}\bar{A}_+ n}}{(\sinh^2\frac{\bar{A}_+ n}{2} + \sin^2\frac{|\bar{A}_-|n}{2})} \quad , \end{aligned} \tag{4.5}$$

where $\bar{A}_+ = A_+/l$ and $|\bar{A}_-| = |A_-|/l$ and the divergent part of the action takes the form

$$\begin{aligned} W_{div}[B_3] &= -\frac{1}{2} \frac{1}{(4\pi)^{3/2}} V_{B_3} \int_{\epsilon^2}^{\infty} \frac{ds}{s^{5/2}} e^{-\mu s} \\ &= -\frac{1}{(4\pi)^{3/2}} V_{B_3}\ (\frac{1}{3\epsilon^3} - \frac{\mu^2}{\epsilon} + \frac{2}{3}\mu^{3/2}\sqrt{\pi} + O(\epsilon)) \quad , \end{aligned} \tag{4.6}$$

where we used (3.22) to carry out the integration over s in (4.5).

Remarkably, the expression (4.5) is invariant under transformation: $|\bar{A}_-| \to |\bar{A}_-| + 2\pi k$. As discussed in [7] this is a consequence of the invariance of B_3 under large diffeomorphisms corresponding to Dehn twists: the identifications $i)$ and $ii)$ determining the geometry of B_3 are unchanged if we replace $r_+ \to r_+ \ , \ \ |r_-| \to |r_-| + kl$ for any integer k. This invariance appears only for the Euclidean black hole and disappears when we make the Lorentzian continuation (see discussion below).

The first quantum correction to the action due to quantization of the three-dimensional gravity itself was discussed in [7]. In this case the correction was shown to be determined by only quantity $2\pi(\sinh^2 \frac{\bar{A}_+}{2} + \sin^2 \frac{|\bar{A}_|}{2})$ related with holonomies of the BTZ instanton.

4.2 BTZ instanton with Conical Singularity

The conical BTZ instanton (B_3^α) is obtained from H_3 by the replacing the identification $i)$ as follows:

$i')$. $\theta \to \theta + 2\pi\alpha$

and not changing the identification $ii)$. For $\alpha \neq 1$ the space B_3^α has a conical singularity at the horizon ($\rho = 0$). The heat kernel on B_3^α is constructed via the heat kernel on the regular instanton B_3 by means of the Sommerfeld formula [32], [33]:

$$K_{B_3^\alpha}(x, x', s) = K_{B_3}(x, x', s) + \frac{1}{4\pi\alpha} \int_\Gamma \cot \frac{w}{2\alpha} \quad K_{B_3}(\theta - \theta' + w, s) \quad dw \quad , \tag{4.7}$$

where K_{B_3} is the heat kernel (4.1). The contour Γ in (4.7) consists of two vertical lines, going from $(-\pi + \imath\infty)$ to $(-\pi - \imath\infty)$ and from $(\pi - \imath\infty)$ to $(\pi + \imath\infty)$ and intersecting the real axis between the poles of the $\cot \frac{w}{2\alpha}$: $-2\pi\alpha$, 0 and 0, $+2\pi\alpha$ respectively. For $\alpha = 1$ the integrand in (4.7) is a 2π-periodic function and the contributions from these two vertical lines (at a fixed distance 2π along the real axis) cancel each other.

Applying (4.7) to the heat kernel (4.4) on B_3 we get

$$\begin{aligned} TrK_{B_3^\alpha} = TrK_{B_3} + (2\pi\alpha)\ (\frac{2\pi r_+}{l})\ \frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}\ \frac{\bar{s}}{2}\ [\ \frac{\imath}{4\pi\alpha} \int_\Gamma \frac{\cot \frac{w}{2\alpha}\ dw}{\sin^2 \frac{w}{2}} \\ + \sum_{n=1}^{\infty} e^{-\frac{\Delta\psi_n^2}{4\bar{s}}} \frac{\imath}{4\pi\alpha} \int_\Gamma \frac{\cot \frac{w}{2\alpha}\ dw}{\sinh^2 \frac{\Delta\psi_n}{2} + \sin^2(\frac{w}{2} + \frac{\pi|r_-|}{l} n)}\ ]\ . \end{aligned} \tag{4.8}$$

for the trace of the heat kernel on B_3^α. Note, that the first term comes from the $n = 0$ term (the direct paths) in the sum (4.1), (4.2) while the other one corresponds to $n \neq 0$

(winding paths). Only the $n=0$ term leads to appearance of UV divergences (if $s\rightarrow 0$). The term due to winding paths ($n\neq 0$) is regular in the limit $s\rightarrow 0$ due to the factor $e^{-\frac{\Delta\psi_n^2}{4\bar{s}}}$.

To analyze (4.8) we shall consider the rotating and non-rotating cases separately.

Non-rotating black hole ($J=0,\ |r_-|=0$)

For this case the contour integrals in (4.8) are calculated as follows (see (A.1), (A.2))

$$\frac{\imath}{4\pi\alpha}\int_\Gamma \frac{\cot\frac{w}{2\alpha}\ dw}{\sin^2\frac{w}{2}}=\frac{1}{3}(\frac{1}{\alpha^2}-1)\equiv 2c_2(\alpha)\quad , \tag{4.9}$$

$$\frac{\imath}{4\pi\alpha}\int_\Gamma \frac{\cot\frac{w}{2\alpha}\ dw}{\sinh^2\frac{\Delta\psi_n}{2}+\sin^2\frac{w}{2}}=\frac{1}{\sinh^2\frac{\Delta\psi_n}{2}}\left(\frac{1}{\alpha}\frac{\tanh\frac{\Delta\psi_n}{2}}{\tanh\frac{\Delta\psi_n}{2\alpha}}-1\right) \tag{4.10}$$

Therefore, taking into account that TrK_{B_3} is given by (4.4) multiplied by α we get for the trace (4.8):

$$TrK_{B_3^\alpha}=\left(\frac{V_{B_3^\alpha}}{l^3}+\frac{A_+}{l}(2\pi\alpha)c_2(\alpha)\ \bar{s}\right)\frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}$$
$$+2\pi\ \frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}\ \frac{A_+}{l}\ \bar{s}\sum_{n=1}^{\infty}\frac{\tanh\frac{\Delta\psi_n}{2}}{\tanh\frac{\Delta\psi_n}{2\alpha}}\ \frac{e^{-\frac{\Delta\psi_n^2}{4\bar{s}}}}{\sinh^2\frac{\Delta\psi_n}{2}}\quad , \tag{4.11}$$

where $\Delta\psi_n=\frac{A_+}{l}n$, $A_+=2\pi r_+$.

Rotating black hole ($J\neq 0,\ |r_-|\neq 0$)

When rotation is present we have for the contour integral in (4.8) (see (A.5)):

$$\frac{\imath}{4\pi\alpha}\int_\Gamma\ \frac{\cot\frac{w}{2\alpha}\ dw}{\sinh^2\frac{\Delta\psi_n}{2}+\sin^2(\frac{w}{2}+\frac{\gamma_n}{2})}$$
$$=\frac{1}{\alpha}\ \frac{\sinh\frac{\Delta\psi_n}{\alpha}}{\sinh\Delta\psi_n}\ \frac{1}{(\sinh^2\frac{\Delta\psi_n}{\alpha}+\sin^2\frac{[\gamma_n]}{2\alpha})}-\frac{1}{(\sinh^2\Delta\psi_n+\sin^2\frac{\gamma_n}{2})} \tag{4.12}$$

where $[\gamma]=\gamma-\pi k,\ |[\gamma]|<\pi$. Then we obtain for the heat kernel on B_3^α:

$$TrK_{B_3^\alpha}=\left(\frac{V_{B_3^\alpha}}{l^3}+\frac{A_+}{l}(2\pi\alpha)c_2(\alpha)\ \bar{s}\right)\frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}$$
$$+2\pi\frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}\frac{A_+}{l}\ \bar{s}\sum_{n=1}^{\infty}\frac{\sinh\frac{\Delta\psi_n}{\alpha}}{\sinh\Delta\psi_n}\ \frac{e^{-\frac{\Delta\psi_n^2}{4\bar{s}}}}{(\sinh^2\frac{\Delta\psi_n}{2\alpha}+\sin^2\frac{[\gamma_n]}{2\alpha})}\quad , \tag{4.13}$$

where $\gamma_n=|A_-|n/l$ and $\Delta\psi_n=A_+n/l$. Remarkably, (4.13) has the periodicity $\gamma_n\rightarrow\gamma_n+2\pi\alpha$ or equivalently $|A_-|n/l\rightarrow|A_-|n/l+2\pi\alpha$.

As discussed in Section 2, any result obtained for the Euclidean black hole must be analytically continued to Lorentzian values of the parameters by means of (2.5). For the

non-rotating black hole this is rather straightforward. It simply means that the area A_+ of the Euclidean horizon becomes the area of the horizon in the Lorentzian space-time. For a rotating black hole the procedure is more subtle. From (2.5) we must also transform $|A_-|$ which after analytic continuation becomes imaginary ($|A_-| \to \imath A_-$), where A_- is area of the lower horizon of the Lorentzian black hole. Doing this continuation in the left hand side of the contour integral (4.12) we find that the right hand side becomes

$$\sin^2(\frac{\imath\gamma_n}{2}) = -\sinh^2\frac{\gamma_n}{2}\ , \quad \sin^2(\frac{[\imath\gamma_n]}{2\alpha}) = -\sinh^2\frac{\gamma_n}{2\alpha}\ \ , \tag{4.14}$$

where $\gamma_n = A_- n/l$. Below we are assuming this kind of substitution when we are applying our formulas to the Lorentzian black hole. We see that after the continuation we lose periodicity with respect to γ_n.

It should be noted that there is only a small group of conical spaces for which the heat kernel is known explicitly [34]. (The small s expansion for the heat kernel on conical spaces has been more widely studied, and a rather general result that the coefficients of this expansion contain terms (additional to the standard ones) due to the conical singularity only and are defined on the singular subspace Σ has recently been obtained [35], [36].) However, no black hole geometry among these special cases were known. In (4.13) we have an exact result for a rather non-trivial example of a black hole with rotation, providing us with an exciting possibility to learn something new about black holes. We consider some of these issues in the context of black hole thermodynamics in the next section.

Small s Expansion of the Heat Kernel

As we can see from Eqs.(4.11), (4.13) the trace of the heat kernel on the conical space B_3^α both for the rotating and non-rotating cases has the form

$$TrK_{B_3^\alpha} = \left(\frac{V_{B_3^\alpha}}{l^3} + \frac{A_+}{l}(2\pi\alpha)c_2(\alpha)\ \bar{s}\right)\frac{e^{-\mu\bar{s}}}{(4\pi\bar{s})^{3/2}}\ +\ ES\ \ , \tag{4.15}$$

where ES stands for exponentially small terms which behave as $e^{-\frac{1}{s}}$ in the limit $s \to 0$. So, for small s we get the asymptotic formula

$$TrK_{B_3^\alpha} = \frac{1}{(4\pi s)^{3/2}}\left(V_{B_3^\alpha} + (-\frac{1}{l^2}V_{B_3^\alpha} + \frac{\xi}{l^2}V_{B_3^\alpha} + A_+\ (2\pi\alpha)c_2(\alpha))\ s\ + O(s^2)\right) \tag{4.16}$$

where $\mu = 1 - \xi$.

The asymptotic behavior of the heat kernel on various manifolds is well known and the asymptotic expressions are derived in terms of geometrical invariants of the manifold. For the operator $(\Box + X)$, where X is some scalar function, on a d-dimensional manifold M^α with conical singularity whose angular deficit is $\delta = 2\pi(1-\alpha)$ at the surface Σ, the corresponding expression reads

$$TrK_{M^\alpha} = \frac{1}{(4\pi s)^{d/2}}(a_0 + a_1 \; s + O(s^2)) \quad , \tag{4.17}$$

where

$$a_0 = \int_{M^\alpha} 1 \quad , \quad a_1 = \int_{M^\alpha} (\frac{1}{6}R + X) \; + \; (2\pi\alpha)c_2(\alpha)\int_\Sigma 1 \quad . \tag{4.18}$$

The volume part of the coefficients is standard [37] while the surface part in a_1 is due to the conical singularity according to [35]. One can see that (4.16) exactly reproduces (4.17)-(4.18) for operator $(\Box + \xi/l^2)$ since for the case under consideration we have $R = -6/l^2$. Note, that in (4.15), (4.16) we do not obtain the usual term $\int_{\partial M} k$ due to extrinsic curvature k of boundary ∂M. This term does not appear in our case since we calculate the heat kernel for spaces with boundary lying at infinity where the boundary term is divergent. But, it would certainly appear if we deal with a boundary staying at a finite distance. Also, in the expressions (4.15), (4.16) we do not observe a contribution due to extrinsic curvature of the horizon surface. According to arguments by Dowker [36] such a contribution to the heat kernel occurs for generic conical space. However, in the case under consideration the extrinsic curvature of the horizon precisely vanishes. We observed [19] the similar phenomenon for charged Kerr black hole in four dimensions.

Effective action and renormalization

For the effective action we immedately obtain that

$$\begin{aligned} W_{eff}[B_3^\alpha] &= -\frac{1}{2}\int_{\epsilon^2}^\infty \frac{ds}{s} TrK_{B^3_\alpha} \\ &= W_{div}[B_3^\alpha] - \sum_{n=1}^\infty \frac{1}{4n} \; \frac{\sinh(\frac{\bar{A}_+}{\alpha}n)}{\sinh(\bar{A}_+ n)} \; \frac{e^{-\sqrt{\mu}\bar{A}_+ n}}{(\sinh^2 \frac{\bar{A}_+ n}{2\alpha} + \sin^2 \frac{[|\bar{A}_-|n]}{2\alpha})} \quad , \end{aligned} \tag{4.19}$$

where the divergent part $W_{div}[B_3^\alpha]$ of the effective action takes the form

$$W_{div}[B_3^\alpha] = -\frac{1}{2}\frac{1}{(4\pi)^{3/2}} \left(V_{B_3^\alpha} \int_{\epsilon^2}^\infty \frac{ds}{s^{5/2}} e^{-\mu s/l^2} \; + \; A_+ \; (2\pi\alpha) \; c_2(\alpha) \int_{\epsilon^2}^\infty \frac{ds}{s^{3/2}} e^{-\mu s/l^2} \right)$$

$$
= -\frac{1}{(4\pi)^{3/2}} \ [V_{B_3^\alpha} \ (\frac{1}{3\epsilon^3} - \frac{\mu}{l^2\epsilon} + \frac{2}{3}\frac{\mu^{3/2}}{l^3}\sqrt{\pi} + O(\epsilon))
$$

$$
+A_+ \ (2\pi\alpha) \ c_2(\alpha)(\frac{1}{\epsilon} - \frac{\sqrt{\mu\pi}}{l} + O(\epsilon))] \quad . \tag{4.20}
$$

Recall that Eqs.(4.19), (4.20) must be analytically continued by means of (2.5) and (4.14) to deal with the characteristics of the Lorentzian black hole. Note that the rotation parameter J enters the UV-infinite part (4.19) only via A_+. The form of (4.19) is therefore the same for rotating and non-rotating holes. Similar behavior for an uncharged Kerr black hole was previously observed in four dimensions [19].

The classical gravitational action

$$
W = -\frac{1}{16\pi G_B}\int_M (R + \frac{2}{l^2}) = -\frac{1}{16\pi G_B}\int_M R - \lambda_B \int_M 1 \quad ,
$$

where $\lambda_B = \frac{1}{8\pi G_B}\frac{1}{l^2}$. In the presence of a conical singularity with angular deficit $\delta = 2\pi(1-\alpha)$ on a surface of area A_+ this has the form

$$
W = -\frac{1}{16\pi G_B}\int_{M^\alpha} R - \frac{1}{4G_B}A_+(1-\alpha) - \lambda_B \int_{M^\alpha} 1 \quad . \tag{4.21}
$$

The $\frac{1}{\epsilon}$ and $\frac{1}{\epsilon^3}$ UV-divergences of the effective action (4.19)-(4.20) for $\alpha = 1$ (regular manifold without conical singularities) are known to be absorbed in the renormalization of respectively the bare Newton constant G_B and cosmological constant λ_B of the classical action. As was pointed out in [16] and [18] the divergences of the effective action that are of first order with respect to $(1-\alpha)$ are automatically removed by the same renormalization of Newton's constant G_B in the classical action (4.21). This statement is important in the context of the renormalization of UV-divergences of the black hole entropy. Its validity in the case under consideration can be easily demonstrated if we note that $(2\pi\alpha)c_2(\alpha) = \frac{2}{3}\pi(1-\alpha) + O((1-\alpha)^2)$ and define the renormalized quantities G_{ren} and λ_{ren} as follows:

$$
\frac{1}{16\pi G_{ren}} = \frac{1}{16\pi G_B} + \frac{1}{12}\ \frac{1}{(4\pi)^{3/2}}\ \int_{\epsilon^2}^{\infty} \frac{ds}{s^{3/2}} e^{-\mu s/l^2} \tag{4.22}
$$

and

$$
\lambda_{ren} = \lambda_B + \frac{1}{2}\ \frac{1}{(4\pi)^{3/2}}\ \left(\int_{\epsilon^2}^{\infty} \frac{ds}{s^{5/2}} e^{-\mu s/l^2} + l^2 \int_{\epsilon^2}^{\infty} \frac{ds}{s^{3/2}} e^{-\mu s/l^2}\right) \quad . \tag{4.23}
$$

Then all the divergences in (4.20) which are up to order $(1-\alpha)$ are renormalized by (4.22), (4.23). The renormalization of terms $\sim O((1-\alpha)^2)$ requires in principle the introduction of some new counterterms. However they are irrelevant for black hole entropy.

The prescription (4.22), (4.23) includes in part some UV-finite renormalization. This is in order that G_{ren} and λ_{ren} be treated as macroscopically measurable constants. Note also that the relation between the bare constants: $\lambda_B G_B = \frac{1}{8\pi}\frac{1}{l^2}$ is no longer valid for the renormalized quantities (4.22)-(4.23).

5 Entropy

A consideration of the conical singularity at the horizon for the Euclidean black hole is a convenient way to obtain the thermodynamic quantities of the hole. Geometrically, the angular deficit $\delta = 2\pi(1-\alpha)$, $\alpha = \frac{\beta}{\beta_H}$ appears when we close the Euclidean time coordinate with an arbitrary period $2\pi\beta$. Physically it means that we consider the statistical ensemble containing a black hole at a temperature $T = (2\pi\beta)^{-1}$ different from the Hawking value $T_H = (2\pi\beta_H)^{-1}$. The state of the system at the Hawking temperature is the equilibrium state corresponding to the extremum of the free energy [24]. The entropy of the black hole appears in this approach as the result of a small deviation from equilibrium. Therefore in some sense the entropy is an off-shell quantity. If $W[\alpha]$ is the action calculated for arbitrary angular deficit δ at the horizon we get

$$S = (\alpha\partial_\alpha - 1)W[\alpha]|_{\alpha=1} \quad . \tag{5.1}$$

for the black hole entropy. Applying this formula to the classical gravitational action (4.21) we obtain the classical Bekenstein-Hawking entropy:

$$S_{BH} = \frac{A_+}{4G_B} \quad . \tag{5.2}$$

Applying (5.1) to the (renormalized) quantum action $W + W_{eff}$ (4.19), (4.21) we obtain the (renormalized) quantum entropy of black hole:

$$\begin{aligned} S &= \frac{A_+}{4G} + \sum_{n=1}^{\infty} s_n \quad , \\ s_n &= \frac{1}{2n}\frac{e^{-\sqrt{\mu}\bar{A}_+ n}}{(\cosh \bar{A}_+ n - \cosh \bar{A}_- n)}(1 + \bar{A}_+ n \coth \bar{A}_+ n \\ &-\frac{(\bar{A}_+ n \sinh \bar{A}_+ n - \bar{A}_- n \sinh \bar{A}_- n)}{(\cosh \bar{A}_+ n - \cosh \bar{A}_- n)}) \quad , \end{aligned} \tag{5.3}$$

where $G \equiv G_{ren}$ is the renormalized Newton constant. We already have done the analytic continuation (4.14) in (5.3) in order to deal with the characteristics of the Lorentzian black hole. The second term in the right hand side of (5.3) can be considered to be the one-loop quantum (UV-finite) correction to the classical entropy of black hole.

Since $\frac{A_-}{A_+} = k < 1$, s_n is a non-negative quantity which monotonically decreases with n and has asymptotes:

$$s_n \to \frac{1}{4n} e^{-(1+\sqrt{\mu})\bar{A}_+ n} \quad if \quad A_+ \to \infty \tag{5.4}$$

and

$$s_n \to \frac{1}{6n} - \frac{\mu}{6}\bar{A}_+ \quad if \quad A_+ \to 0 \quad . \tag{5.5}$$

Note that both asymptotes (5.4), (5.5) are independent of the parameter A_- characterizing the rotation of the hole.

The infinite sum in (5.3) can be approximated by integral. We find that

$$S = \frac{A_+}{4G} + \int_{\bar{A}_+}^{\infty} s(x)\ dx \quad , \tag{5.6}$$

where

$$s(x) = \frac{1}{2x} \frac{e^{-\sqrt{\mu}x}}{(\cosh x - \cosh kx)} \left(1 + x\coth x - \frac{(x\sinh x - kx\sinh kx)}{(\cosh x - \cosh kx)}\right) \quad . \tag{5.7}$$

For large enough $\bar{A}_+ \equiv \frac{A_+}{l} >> 1$ the integral in (5.6) exponentially goes to zero and we have the classical Bekenstein-Hawking formula for entropy. On the other hand, for small $\bar{A}_+$ the integral in (5.6) is logarithmically divergent so that we have

$$S = \frac{A_+}{4G} + \frac{\sqrt{\mu}}{6}\frac{A_+}{l} - \frac{1}{6}\ln\frac{A_+}{l} + O((\frac{A_+}{l})^2) \quad . \tag{5.8}$$

This logarithmic divergence can also be understood by examining the expression (5.3). From (5.5) it follows that every n-mode gives a finite contribution $s_n = \frac{1}{6n}$ at zero A_+. Their sum $\sum_{n=1}^{\infty} \frac{1}{6n}$, however, is not convergent since s_n does not decrease fast enough. This divergence appears as the logarithmic one in (5.8). This logarithmic behavior for small A_+ is universal, independent of the constant ξ (or μ) in the field operator and the area of the inner horizon (A_-) of the black hole. Hence the rotation parameter J enters (5.8) only via the area A_+ of the larger horizon. It should be note that similar logarithmic

behavior was previously observed in various models both in two [16], [24] and four [25], [26] dimensions. Remarkably, it appears in the three dimensional model as the result of an explicit one-loop calculation.

The first quantum correction to the Bekenstein-Hawking entropy due to quantization of the three-dimensional gravity itself was calculated in [7] and was shown to be proportional to area A_+ of outer horizon.

6 Concluding Remarks

Our computation of the quantum-corrected entropy (5.6) of the BTZ black hole has yielded the interesting result that the entropy is not proportional to the outer horizon area (*i.e.* circumference) A_+, but instead develops a minimum for sufficiently small A_+. (The plot of the entropy as function of area A_+ for non-rotating case is represented in Fig.2.) This minimum is a solution to the equation

$$\frac{l}{4G} = s(\frac{A_{+min}}{l}) \quad . \tag{6.1}$$

The constants G and l determine two different scales in the theory. The former determines the strength of the gravitational interaction. The distance $l_{pl} \sim G$ can be interpreted as the Planck scale in this theory. It determines the microscopic behavior of quantum gravitational fluctuations. On the other hand, the constant l (related to curvature via $R = -6/l^2$) can be interpreted as radius of the Universe that contains the black hole. So l is a large distance (cosmological) scale.

Regardless of the relative sizes of G and l, the entropy is always minimized for $A_+ \leq G$. If we assume $G << l$ then (6.1) is solved as $A_{+min} = \frac{2}{3}G$, However if $G >> l$, then (6.1) becomes (for $\mu = 0$, say) $A_+/G \simeq e^{-A_+/l} < 1$. In either case, the minimum of the entropy occurs for a hole whose horizon area is of the order of the Planck length $r_+ \sim l_{pl}$. In the process of evaporation the horizon area of a hole typically shrinks. The evaporation is expected to stop when the black hole takes the minimum entropy configuration. In our case it is the configuration with horizon area $A_+ = A_{+min}$. Presumably it has zero temperature and its geometry is a reminscent of an extremal black hole. However at present we cannot definitively conclude this since our considerations do not take into

account quantum back reaction effects. These effects are supposed to drastically change the geometry at a distance $r \sim l_{pl}$. Therefore the minimum entropy configuration is likely to have little in common with the classical black hole configuration described in Section 2. Further investigation of this issue will necessitate taking the back reaction into account.

Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council of Canada and by a NATO Science Fellowship.

Appendix: Contour integrals

Consider the integral

$$I = \frac{1}{2\pi\imath} \int_\Gamma f(w)\, dw \quad ,$$
$$f(w) = \cot\frac{w}{2\alpha}\, \frac{1}{a^2 + \sin^2(\frac{w+\gamma}{2})} \quad , \tag{A.1}$$

where Im a = Im $\gamma = 0$ and the contour Γ in (A.1) consists of two vertical lines, going from $(-\pi + \imath\infty)$ to $(-\pi - \imath\infty)$ and from $(\pi - \imath\infty)$ to $(\pi + \imath\infty)$ and intersecting the real axis between the poles of the $\cot\frac{w}{2\alpha}$: $-2\pi\alpha,\ 0$ and $0,\ +2\pi\alpha$ respectively.

The integration in (A.1) is carried out by calculating the residues of the function $f(w)$. Let us assume that $-\pi < \gamma < \pi$ ($|\gamma| < \pi$). Then the function $f(w)$ has the following poles and residues:

a). $w = w_0 = 0$,

$$Res\ f(w_0) = \frac{2\alpha}{a^2 + \sin^2\frac{\gamma}{2}}$$

b). $w = w_\pm = -\gamma \pm 2\imath A$,

$$Res\ f(w_\pm) = \frac{2\imath}{\sinh 2A} \left(\pm\frac{\tan\frac{\gamma}{2\alpha}}{\cosh^2\frac{A}{\alpha}} + \frac{\imath\tanh\frac{A}{\alpha}}{\cos^2\frac{\gamma}{2\alpha}} \right) \left(\tanh^2\frac{A}{\alpha} + \tan^2\frac{\gamma}{2\alpha} \right)^{-1} \quad ,$$

where we have introduced A related with a as follows: $\sinh A = a$.

Then the integral (A.1) reads

$$I = \sum_{w=w_0,w_+,w_-} Res\ f(w)$$
$$= \left(\frac{2\alpha}{\sinh^2 A + \sin^2\frac{\gamma}{2}} - \frac{\sinh\frac{2A}{\alpha}}{\sinh 2A}\, \frac{2}{(\sinh^2\frac{A}{\alpha} + \sin^2\frac{\gamma}{2\alpha})} \right) \quad . \tag{A.2}$$

Two special cases of this expression are worth noting. If $\gamma = 0$ we get from (A.2):

$$I = \frac{2}{\sinh^2 A}(\alpha - \frac{\tan A}{\tan\frac{A}{\alpha}}) \tag{A.3}$$

and if $\gamma = a = 0$ we get (see also [35])

$$I = \frac{2}{3}(\alpha - \frac{1}{\alpha}) \equiv -4\alpha c_2(\alpha) \tag{A.4}$$

If $\pi < |\gamma| < 2\pi$ the structure of the pole at $w = w_0 = 0$ remains the same as above while the other poles lying in the region $-\pi < w < \pi$ are $w_\pm = 2\pi - \gamma \pm 2\imath A$. Hence the corresponding residue takes the same form b), with the replacment $\gamma \to 2\pi - \gamma$. Next, taking γ to be arbitrary, define $[\gamma]$ as follows: $|\gamma| = \pi k + [\gamma]\ , [\gamma] < \pi$. Then for arbitrary γ the integral (A.1) reads

$$I = 2\left(\frac{\alpha}{\sinh^2 A + \sin^2\frac{\gamma}{2}} - \frac{\sinh\frac{2A}{\alpha}}{\sinh 2A}\,\frac{1}{(\sinh^2\frac{A}{\alpha} + \sin^2\frac{[\gamma]}{2\alpha})}\right)\quad . \tag{A.5}$$

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Figure Captions

Fig.1 A section of BTZ black hole at fixed χ for the non-rotating ($|r_-| = 0$) and rotating cases. The opposite sides of the quadrangles are identified. Therefore, the whole section looks like a torus. In the rotating case the torus is deformed with deformation parameter γ, where $\tan\gamma = \frac{r_+}{|r_-|}$. Both the quadrangles have the same area.

Fig.2 The plot of quantum entropy vs. area for the non-rotating case and $\mu = 0$, $G = l = 1$.

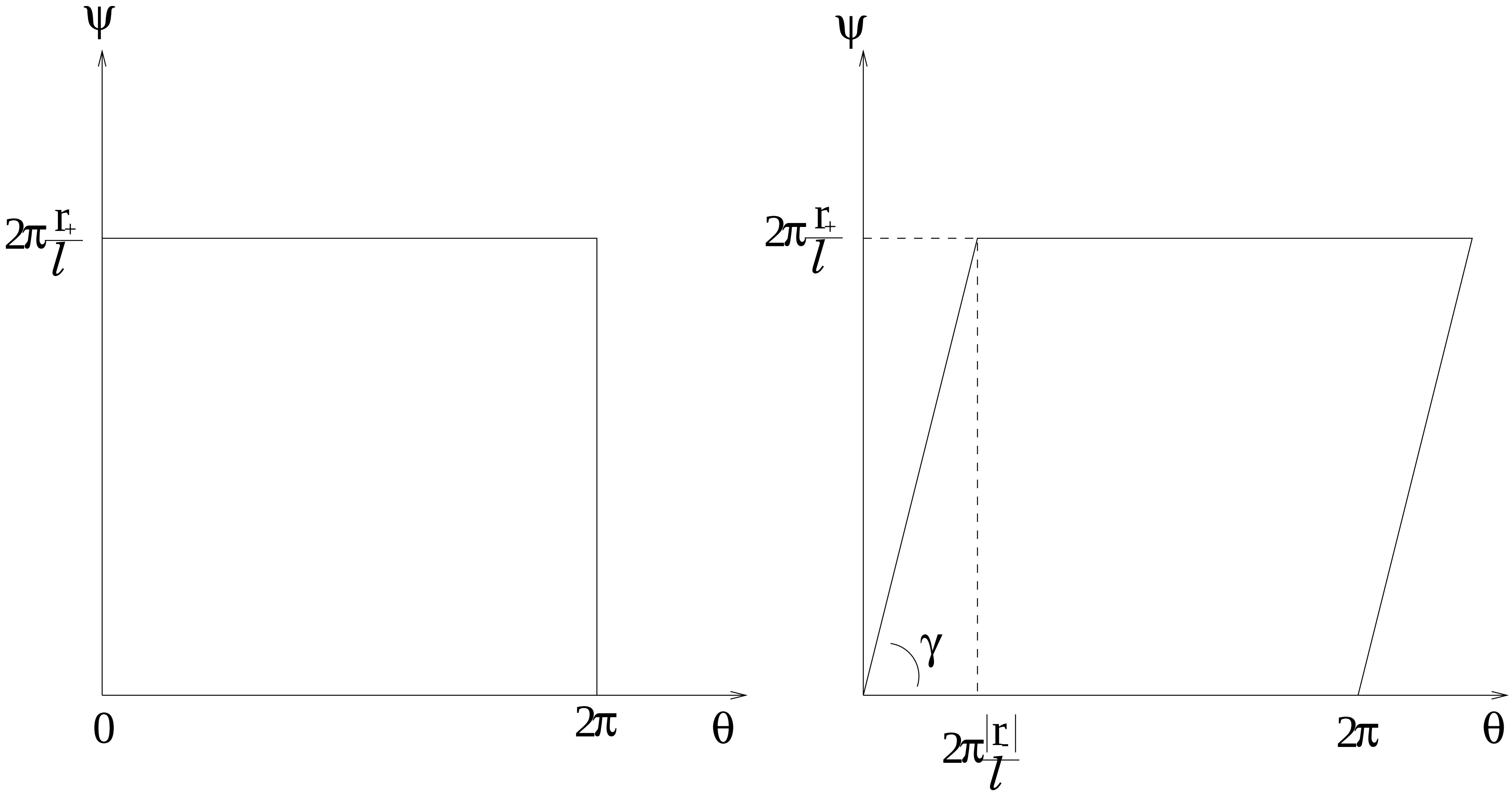

Fig. 1

Fig. 2